Name: RED ID:

Fall 2021 Math 245 Final Exam

Please read the following directions:

Please write legibly, with plenty of white space. Please **print** your name and REDID in the designated spaces above. Please fit your answers into the designated areas; material outside the designated areas (such as on this cover page) will not be graded. To get credit, you must also show adequate work to justify your answers. If unsure, show the work. Each of the twenty problems is worth 8-16 points. The use of notes, books, calculators, or other materials on this exam is strictly prohibited, except you may bring one 8.5"x11" page (both sides) with your handwritten notes. This exam will begin at 10:30 and will end at 12:30; pace yourself accordingly. Good luck!

Special exam instructions for HH-130:

1. Please stow all bags/backpacks/boards at the front or rear of the room. All contraband, except phones, must be stowed in your bag. All phones must be silent, non-vibrating, and either in your pocket or stowed in your bag.
2. Please remain quiet to ensure a good test environment for others.
3. Please keep your exam on your desk; do not lift it up for a better look.
4. If you have a question or need the restroom, please come to the front. Bring your exam.
5. If you are done and want to submit your exam and leave, please wait until one of the designated exit times, listed below. Please do **NOT** leave at any other time. If you are sure you are done, just sit and wait until the next exit time, with this cover sheet visible.

Designated exam exit times:
10:50 "See you next semester"
11:10 "I wish I had studied more"
11:30 "One extra hour of drinking – worth it"
11:50 "Maybe this will be good enough"
12:10 "There is nothing more in my brain, let me out of here"
12:30 "I need every second I can get"

REMINDER: Use complete sentences.

Problem 1. Carefully state the following definitions/theorems:

a. prime

b. binomial coefficient

Problem 2. Carefully state the following definitions/theorems:

a. implies

b. cardinality

Problem 3. Carefully state the following definitions/theorems:

a. equivalence relation

b. right-definite

Problem 4. Carefully state the following definitions/theorems:

a. associativity theorem (for propositions)

b. The Master Theorem

Problem 5. Let p, q, r be propositions. Without using truth tables, prove:
$$(\neg r) \to (\neg p), r \to q \vdash p \to q.$$

Problem 6. Prove or disprove: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R},\ x < y < \lfloor x \rfloor + 1$.

Problem 7. Let $x \in \mathbb{R}$. Prove that $\lfloor x \rfloor$ is unique.

Problem 8. Prove that $\forall n \in \mathbb{N}_0,\ \sum_{k=0}^{n} k2^k = 2 + (n-1)2^{n+1}$.

Problem 9. Let S, U be sets, with $S \subseteq U$. Prove that $U \subseteq S \cup S^c$.

Problem 10. Prove or disprove: For all sets S, T, $2^{S \cap T} \subseteq 2^S \cap 2^T$.

Problem 11. Prove or disprove: For all sets S, T with $S \subseteq T$, and for all relations R on S, R must be a relation from S to T.

Problem 12. Let a_n, b_n be sequences. Set $c_n = 100a_n$. Prove: if $a_n = O(b_n)$, then $c_n = O(b_n)$.

Problem 13. Define relation R on $\mathbb{Z}$ via $R = \{(a,b) : |a-b| \leq 17\}$. Prove that R is *NOT* an equivalence relation.

Problem 14. Find all integers $x \in [0, 100)$ satisfying $15x \equiv 50 \pmod{100}$.

Problem 15. Consider equivalence relation R on $\mathbb{Z}$ via $R = \{(a,b) : (a^2-5)^2 = (b^2-5)^2\}$. Compute the equivalence class $[1]_R$.

Problem 16. Let R be a partial order on S. Suppose that $a \in S$ is both maximal and minimal in S. Prove: $\forall b \in S,\ (b \| a) \vee (b = a)$.

Problem 17. Let R be the partial order $\mid$ on $\mathbb{N}$, and set R^2 to be the product order on $\mathbb{N} \times \mathbb{N}$ that we get from combining R with itself. Draw the Hasse diagram for the interval poset, within R^2, given by $[(1,1),(6,7)]$.

Problem 18. Consider the relation $R : \mathbb{N} \to \mathbb{N}$ given by $\{(x,y) : (x-2)^2 = (y-2)^2\}$. Determine, with proof, whether R is a well-defined function.

Problem 19. Let R, S, T be sets, and suppose that $F_1 : R \to S$ and $F_2 : S \to T$ are functions. Prove that $F_2 \circ F_1$ is a well-defined function.

Problem 20. Set $S = \{x \in \mathbb{R} : x > 0\}$. Consider the function $f : \mathbb{R} \to S$ given by $f(x) = e^{-x/2}$. Prove that $f(x)$ is a bijection.